The Equations Defining Rees Algebras of Ideals of Hypersurface Rings

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Background

Suppose R is a Noetherian ring and I an R-ideal. Recall the Rees algebra is the graded ring

$$\mathcal{R}(I) = R \oplus It \oplus I^2t^2 \oplus \cdots = R[It] \subset R[t]$$

- If we specify a generating set, say $I = (\alpha_1, \ldots, \alpha_n)$, there is a natural epimorphism

$$\Psi : \, R[T_1, \ldots, T_n] \longrightarrow \mathcal{R}(I)$$

  where $\Psi(T_i) = \alpha_i t$.

- The kernel of this map $\mathcal{J} = \ker \Psi$ is the *defining ideal* of $\mathcal{R}(I)$.

- Minimal generators of $\mathcal{J}$ are the *defining equations* of $\mathcal{R}(I)$.

The Symmetric Algebra

The original map factors through the symmetric algebra:

$$\begin{array}{ccc} R[T_1,\ldots,T_n] & \xrightarrow{\quad\Psi\quad} & \mathcal{R}(I) \\ & {\scriptstyle\sigma}\searrow \quad \nearrow & \\ & \mathrm{Sym}(I) & \end{array}$$

- If $R^m \xrightarrow{\varphi} R^n \longrightarrow I \longrightarrow 0$ is a presentation of I, $\mathcal{L} = \ker\sigma$ is generated by the entries of the matrix product

$$[T_1 \ldots T_n] \cdot \varphi = [\ell_1 \ldots \ell_m]$$

- With this $\mathcal{L} \subseteq \mathcal{J}$.
- Often $\mathcal{L} \subsetneq \mathcal{J}$, but if they are equal I is of *linear type*.

The G_s Condition

Definition

An R-ideal I is said to satisfy G_s if $\mu(I_\mathfrak{p}) \leq \dim R_\mathfrak{p}$ for all $\mathfrak{p} \in V(I)$ with $\dim R_\mathfrak{p} \leq s-1$. If I satisfies G_s for all s, then I is said to satisfy G_∞.

- Equivalently I satisfies G_s if and only if $\operatorname{ht} \operatorname{Fitt}_j(I) \geq j+1$ for all $1 \leq j \leq s-1$.
- Recall

$$\dim \operatorname{Sym}(I) = \sup \left\{ \mu(I_\mathfrak{p}) + \dim R/\mathfrak{p} \,\middle|\, \mathfrak{p} \in \operatorname{Spec}(R) \right\}$$

- If I satisfies G_s for large s, $\mathcal{L}$ is “closer” to $\mathcal{J}$

The Jacobian Dual

- With $R^m \xrightarrow{\varphi} R^n \longrightarrow I \longrightarrow 0$ a presentation as before, suppose $I_1(\varphi) = (x_1, \ldots, x_s)$

- There exists an $s \times m$ matrix $B(\varphi)$ such that

$$[T_1, \ldots, T_n] \cdot \varphi = [x_1, \ldots, x_s] \cdot B(\varphi)$$

- $B(\varphi)$ is a *Jacobian dual* of φ.

- By Cramer's rule $I_s\big(B(\varphi)\big) \subset \mathcal{J}$

- $B(\varphi)$ is not necessarily unique!

- If R is standard graded and φ consists of linear entries, it is unique.

A Known Result

Theorem (Morey & Ulrich, 1996)

If $R = k[x_1, \ldots, x_d]$ and I is a linearly presented perfect R-ideal of grade 2 satisfying G_d, then $\mathcal{J} = \mathcal{L} + I_d\big(B(\varphi)\big)$ where φ is the Hilbert-Burch presentation matrix for I. Furthermore $\mathcal{R}(I)$ is Cohen-Macaulay.

- One says that $\mathcal{L} + I_d\big(B(\varphi)\big)$ is the *expected form* of $\mathcal{J}$.
- It is interesting to ask if $\mathcal{J}$ is of the expected form if any of the assumptions are weakened:
  - G_{d-1} — Nguyen (2014)
  - Almost linear presentation — Boswell & Mukundan (2016)
  - Hypersurface ring — Weaver (2021)

Ideals in Hypersurface Rings

- $S = k[x_1, \ldots, x_{d+1}]$, $f \in S$ a homogeneous polynomial of degree m, and $R = S/(f)$
- I a grade 2 perfect R-ideal satisfying G_d
- φ the Hilbert-Burch presentation matrix of I consisting of linear entries
- Assume $I_1(\varphi) = \overline{(x_1, \ldots, x_{d+1})}$ and $\mu(I) = d+1$
- If $\mathcal{J}$ is the defining ideal of $\mathcal{R}(I)$, is $\mathcal{J} = \mathcal{L} + I_{d+1}\big(B(\varphi)\big)$? No!
- Notice $B(\varphi)$ is $(d+1) \times d$

Back to $S = k[x_1, \dots, x_{d+1}]$

- Let $\overline{\cdot}$ denote images modulo (f)
- Let ψ be a $(d+1) \times d$ matrix of linear entries of S with $\varphi = \overline{\psi}$
- There is a grade 2 perfect S-ideal J with ψ as its presentation matrix!
  - By Hilbert-Burch Theorem
- $I_1(\psi) = (x_1, \dots, x_{d+1})$
- J satisfies ~~G_d~~, ~~G_{d+1}~~, G_∞ (hence linear type)
- We now know the defining ideal of $\mathcal{R}(J)$!

Defining Ideals of Sym(I)

Let $[T_1, \ldots, T_{d+1}] \cdot \psi = [\ell_1, \ldots, \ell_d]$ and consider the $S[T_1, \ldots, T_{d+1}]$-ideal

$$\mathcal{L} = (\ell_1, \ldots, \ell_d, f)$$

- $\overline{\mathcal{L}}$ is the defining ideal of Sym(I)
- Moreover, $\frac{S[T_1,\ldots,T_{d+1}]}{\mathcal{L}} \cong \frac{R[T_1,\ldots,T_{d+1}]}{\overline{\mathcal{L}}} \cong \mathrm{Sym}(I)$
- $\mathcal{L}$ is also a defining ideal of Sym(I), but as an S-algebra!
- Is there a similar analogous ideal in $S[T_1, \ldots, T_{d+1}]$ for the defining ideal of $\mathcal{R}(I)$?

Defining Ideals of $\mathcal{R}(I)$

Let $\mathcal{J}$ denote the defining ideal of $\mathcal{R}(I)$ and recall $\overline{\mathcal{L}}$ is the defining ideal of $\text{Sym}(I)$

- I satisfies G_d so it is linear type *on the punctured spectrum* of R.

$$\text{Sym}(I_\mathfrak{p}) \cong \mathcal{R}(I_\mathfrak{p}) \quad \text{for all } \mathfrak{p} \in \text{Spec}(R) \setminus \{\overline{(x_1, \ldots, x_{d+1})}\}$$

- In other words, $\overline{\mathcal{L}}_\mathfrak{p} \cong \mathcal{J}_\mathfrak{p}$ for these primes
- It then follows that $\mathcal{J} = \overline{\mathcal{L}} : \overline{(x_1, \ldots, x_{d+1})}^\infty$
- Consider the $S[T_1, \ldots, T_{d+1}]$-ideal

$$\mathcal{A} = \mathcal{L} : (x_1, \ldots, x_{d+1})^\infty$$

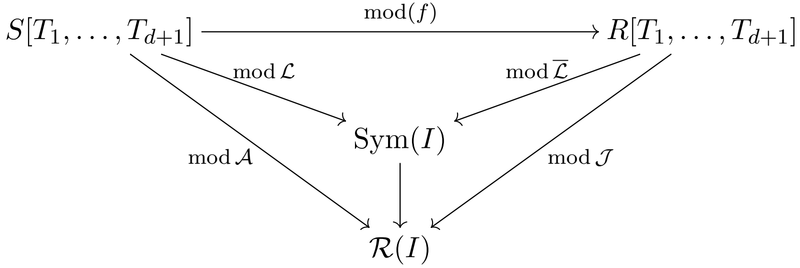
R vs. S

We have produced the following correspondence between objects in R and S

R & $R[T_1,\ldots,T_{d+1}]$	S & $S[T_1,\ldots,T_{d+1}]$
φ	ψ
I	J
G_d	G_{d+1} (G_∞)
$\overline{\mathcal{L}}$	$\mathcal{L}$
$\mathcal{J}$	$\mathcal{A}$

We can also view these connections in the following diagram

R vs. S (Cont.)


$S[T_1, \ldots, T_{d+1}]$
$\text{mod}(f)$
$R[T_1, \ldots, T_{d+1}]$
$\text{mod}\, \mathcal{L}$
$\text{mod}\, \overline{\mathcal{L}}$
$\text{Sym}(I)$
$\text{mod}\, \mathcal{A}$
$\text{mod}\, \mathcal{J}$
$\mathcal{R}(I)$

Saturations

As before, $\mathcal{A} = \mathcal{L} : (x_1, \ldots, x_{d+1})^\infty$

- Recall $\mathcal{L} = (\ell_1, \ldots, \ell_d, f)$ and notice the equations of $\mathcal{L}$ are linear with respect to $x_1, \ldots, x_{d+1}$ except f.
- Similar to Boswell and Mukundan's situation
- Using degree arguments (or local cohomology)

$$\mathcal{A} = \mathcal{L} : (x_1, \ldots, x_{d+1})^m$$

  where $m = \deg f$

- We associate a matrix to $\mathcal{L}$ and the sequence $x_1, \ldots, x_{d+1}$.

Modified Jacobian Dual

Recall $\mathcal{L} = (\ell_1, \ldots, \ell_d, f)$ where $[\ell_1 \ldots \ell_d] = [x_1 \ldots x_{d+1}] \cdot B(\psi)$

- $B(\psi)$ doesn't "see" f
- Let ∂f be any column with $f = [x_1 \ldots x_{d+1}] \cdot \partial f$
- Extend $B(\psi)$ to $\mathcal{B} = [B(\psi) \,|\, \partial f]$
- $\mathcal{B}$ is a *modified Jacobian dual* of ψ
- May take ∂f to be the column of partial derivatives of f if $\operatorname{char} k = 0$
- Notice $\mathcal{B}$ is $(d+1) \times (d+1)$

The Equations of $\mathcal{A} = \mathcal{L} : (x_1, \dots, x_{d+1})^m$

- $\det \mathcal{B} \in \mathcal{A}$ by Cramer's rule
- Is $\mathcal{A} = \mathcal{L} + (\det \mathcal{B})$? No! (unless $m = 1$)
- $\mathcal{L} + (\det \mathcal{B}) = \mathcal{L} : (x_1, \dots, x_{d+1}) \subset \mathcal{A}$
- Idea to get new equations:

$$\mathcal{L} : (x_1, \dots, x_{d+1})^2 = \underbrace{\big(\mathcal{L} : (x_1, \dots, x_{d+1})\big)}_{\mathcal{L}+(\det \mathcal{B})} : (x_1, \dots, x_{d+1})$$

- Can we associate a new matrix to $\mathcal{L} + (\det \mathcal{B})$ and the sequence $x_1, \dots, x_{d+1}$?

Modified Jacobian Dual Iterations

Let $\partial\underline{\cdot}$ denote a column corresponding to an element and the sequence $x_1, \ldots, x_{d+1}$. Recall $m = \deg f$.

$$\mathcal{L}_1 = \mathcal{L} \qquad \mathcal{B}_1 = B(\psi)\,|\,\partial f \qquad F_1 = \det \mathcal{B}_1$$

$$\mathcal{L}_2 = \mathcal{L}_1 + (F_1) \qquad \mathcal{B}_2 = B(\psi)\,|\,\partial F_1 \qquad F_2 = \det \mathcal{B}_2$$

$$\mathcal{L}_3 = \mathcal{L}_2 + (F_2) \qquad \mathcal{B}_3 = B(\psi)\,|\,\partial F_2 \qquad F_3 = \det \mathcal{B}_3$$

$$\vdots \qquad \vdots$$

$$\mathcal{L}_m = \mathcal{L}_{m-1} + (F_{m-1}) \qquad \mathcal{B}_m = B(\psi)\,|\,\partial F_{m-1}$$

By Cramer's rule $\mathcal{L}_i + (\det \mathcal{B}_i) \subseteq \mathcal{L} : (x_1, \ldots, x_{d+1})^i$

A Generating Set of $\mathcal{A}$

- Is $\mathcal{A} = \mathcal{L}_m + (\det \mathcal{B}_m)$? Yes!
- This is actually a *minimal* generating set!
- Adopt the bigrading on $S[T_1, \ldots, T_{d+1}]$

$$\deg x_j = (1,0) \qquad \deg T_j = (0,1)$$

- $\deg \big(\det \mathcal{B}_i\big) = (m-i, d \cdot i)$
- In particular $\mu(\mathcal{A}) = d+m+1$ and $\mu(\mathcal{J}) = d+m$

Cohen-Macaulayness of $\mathcal{R}(I)$

- Recall $\dim \mathcal{R}(I) = d+1$
- $\operatorname{depth} \mathcal{R}(I) \geq d$
  - If $m = 1$, then $\operatorname{depth} \mathcal{R}(I) = d+1$
  - If $m \geq 2$, then $\operatorname{depth} \mathcal{R}(I) = d$
- $\mathcal{R}(I)$ is Cohen-Macaulay if and only if $\deg f = 1$ and is *almost* Cohen-Macaulay otherwise
- Other invariants such as regularity, relation type, etc. can also be deduced

The Main Result

Theorem

Let $S = k[x_1, \dots, x_{d+1}]$, $f \in S$ a homogeneous polynomial of degree m, and $R = S/(f)$. Let I be a perfect R-ideal of grade 2 with linear presentation matrix φ. If I satisfies G_d, $I_1(\varphi) = \overline{(x_1, \dots, x_{d+1})}$, and $\mu(I) = d+1$, then the defining ideal of $\mathcal{R}(I)$ satisfies

$$\mathcal{J} = \overline{\mathcal{L}_m + (\det(\mathcal{B}_m))}$$

where $(\mathcal{L}_m, \mathcal{B}_m)$ is an m^{th} modified Jacobian dual iteration of $(\mathcal{L}, \mathcal{B})$ for modified Jacobian dual $\mathcal{B}$ and $\mathcal{L} = (\underline{x} \cdot \mathcal{B})$. Additionally $\mathcal{R}(I)$ is Cohen-Macaulay if and only if $m = 1$ and is almost Cohen-Macaulay otherwise.

This recovers Morey and Ulrich's result (for $\mu(I) = d+1$) when $\deg f = 1$.

Remarks

- This method fails for $\mu(I) > d+1$
- Using recent work of A. Costantini, this result can be extended to Rees algebras of modules of projective dimension 1 over hypersurface rings.
- A similar process is used to produced the defining equations of the Rees algebra of a perfect Gorenstein ideal of grade 3 of a hypersurface ring
- One could consider Rees algebras of ideals of complete intersection rings